Operator-theoretic view point of Koopman and Frobenius-Perron operators in Dynamical Systems for Control: Literature Review

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Abstract

This paper describes theoretical developments leading to measure theory and the background for the ergodic operator-theoretic approach to linear and nonlinear dynamical systems. The final section of the paper describes current developments on applying the methods to control problems.

Introduction

The Problem

In the 20th century, Ulam, Von Neumann, and many other mathematicians studied this simple systems capable of generating a density of states that has the following quadratic map:

$$S(x) = \alpha x(1-x), \quad \text{for } 0 \leq x \leq 1 \tag{1}$$

When $\alpha = 4$, then S maps onto itself $S : [0,1] \to [0,1]$. The state space of the system is defined to be $[0,1]$ for this case. We can define the trajectory, given an initial point $x^0 \in [0,1]$ and its successive states to be

$$x^0, S(x^0), S(S(x^0)), ... \tag{2}$$

The trajectories of this system was found to be erratic and chaotic for almost all x^0 [1]. The trajectories are also highly sensitive to initial conditions. It was also found that for certain select intial conditions, there exist a point x_* that satisfies

$$x_* = S(x_*) \tag{3}$$

When this phenomena happens, the trajectory will have the constant value x_* forever. There are also other special phenomena that could happen i.e. trajectory becoming periodic for special initial states. The problem is there is no prior way of predicting which initial states will lead to these special phenomenas.

Instead of studying point-wise trajectories that gave us inconclusive results, we can instead study the flow of densities (measures).

Suppose we pick a large number N of intial states and apply the map S.

$$\mathbf{x}_0 = x_1^0, x_2^0, \dots, x_N^0 \tag{4}$$

$$\mathbf{x}_1 = S(\mathbf{x}_0) \tag{5}$$

Then following holds (measure preserving) for density function f_i and $A \subset [0,1]$, where $A \in \mathcal{A}$ in measure space $(X, \mathcal{A}, \mu)$.

$$\int_A f_1(u)du = \int_{S^{-1}(A)} f_0(u)du \tag{6}$$

If A is an interval $A = [a, x]$, then we can obtain an explicit representation for f_1.

$$f_1(x) = \frac{d}{dx} \int_{S^{-1}([a,x])} f_0(u)du = Pf_0(u) \tag{7}$$

Where P is defined as the Frobenius-Perron operator and f_0 is an arbitrary function. We can apply this theory to our problem where $A = [0, x]$.

$$S^{-1}([0,x]) = \left[0, \frac{1}{2} - \frac{1}{2}\sqrt{1-x}\right] \cup \left[\frac{1}{2} + \frac{1}{2}\sqrt{1-x}, 1\right] \tag{8}$$

$$Pf(x) = \frac{1}{4\sqrt{1-x}} \left[f\left(\frac{1}{2} - \frac{1}{2}\sqrt{1-x}\right) + f\left(\frac{1}{2} + \frac{1}{2}\sqrt{1-x}\right)\right] \tag{9}$$

It can be found that as $n \to \infty$ the equation $P^n f$ approaches a unique limiting density f_* such that $Pf_* \equiv f_*$. In this problem, f_* is given by

$$f_*(x) = \frac{1}{\pi\sqrt{x(1-x)}} \tag{10}$$

The limiting density is found to describe the frequency of with which states along a trajectory fall into given regions of the state space. This therefore solves our previous problems.
This problem is in a large part taken from [1] and many details are omitted for conciseness.

The Frobenius-Perron Operator

Let us state a few definitions before defining the Frobenius-Perron Operator.
Let $(X, \mathcal{A}, \mu)$ be a measure space and $S : X \to X$ be a transformation. If $S^{-1}(A) \in \mathcal{A}$ for every interval $A \subset R$, then S is measurable. If $\mu\left(S^{-1}(A)\right) = 0$ for all $A \in \mathcal{A}$ such that $\mu(A) = 0$, then S is nonsingular. If S is a measurable transformation and $\mu\left(S^{-1}(A)\right) = \mu(A)$ for all $A \in \mathcal{A}$, then S is measure preserving or in other words the measure μ is invariant under S.
If S is a nonsingular transformation, the Frobenius-Perron operator is the unique operator $P : L^1 \to L^1$ defined by

$$\int_A Pf(x)\mu(dx) = \int_{S^{-1}(A)} f(x)\mu(dx), \quad \text{for } A \in \mathcal{A} \tag{11}$$

The Frobenius-Perron operator P describes the evolution of f by a transformation S [1].

The Koopman Operator

Let $(X, \mathcal{A}, \mu)$ be a measure space, $S : X \to X$, and $f \in L^\infty$. The Koopman operator with respect to S is the operator $U : L^\infty \to L^\infty$ defined by

$$Uf(x) = f(S(x)) \tag{12}$$

It can easily be seen that the Koopman Operator is closely related to Frobenius-Perron operator, differing only by the function space f is defined in. The Koopman operator is also known to be the adjoint or dual of the Frobenius-Perron operator. Where the following holds for every $f \in L^1$ and $g \in L^\infty$

$$\langle Pf, g\rangle = \langle f, Ug\rangle \tag{13}$$

More properties about the Koopman operator can be found in [12].

Measure Theory

The phenomena we have seen in first part can also be seen as an example of the Poincaré's Recurrence Theorem: If S is measure preserving. Then for any set $A \in \mathcal{A}$ with $\mu(A) > 0$, almost all points of A return infinitely often to A under positive iteration by S [19].

Let us define another important property called ergodicity, this property lies at the core of our approach.

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $S : X \to X$ be a non-singular transformation, then S is ergodic if every invariant set $A \in \mathcal{A}$ are trivial subsets of X i.e. $\mu(A) = 0$

or $\mu(X\backslash A) = 0$. Interesting theorems that arises from this property can be stated as follows.

For every measurable function $f : X \to \mathbb{R}$, S is ergodic if and only if it satisfies 14.

$$f(S(x)) = f(x), \quad \text{for almost all } x \in X \tag{14}$$

This theorem implies that f is constant almost everywhere.

It follows that, if all the fixed points (for some $f \in L^1$, $Uf = f$) of the Koopman operator U are constant functions, then S is ergodic. A similar statement can be said for Frobenius-Perron operators: If S is ergodic, then there is at most one stationary density f_* ($Pf = f$) of P and $f_*(x) > 0$. Or, if there is a unique stationary density f_* of P and $f_*(x) > 0$, then S is ergodic.

Now, let us define the cornerstone of our operator-theoretic approach: the Birkhoff Ergodic Theorem (pointwise convergence) 17 [5] [19] [1].

Let $S : X \to X$ be a measurable transformation and $f : X \to \mathbb{R}$ an integrable function. If the measure μ is invariant, then there exist an integrable function f^* such that

$$f^*(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(S^k(x)\right) \quad \text{for almost all } x \in X \tag{15}$$

This implies that $f^*(x) = f^*(S(x))$ for almost all $x \in X$. If $\mu(X) < \infty$, it can be shown that $\int_X f^*(x)\mu(dx) = \int_X f(x)\mu(dx)$.

This theorem in other words, proves for pointwise convergence of using Lebesgue convergence. It is natural that we want to extend and generalise the Birkhoff's ergodic theorem to more function spaces. The ergodic theorem of Von Neumann 16 help us extend this by proving for convergence in mean in L^p space (Hilbert Spaces) [19].

If S is a measure preserving tranformation and $f \in L^p$ for $1 \le p < \infty$ then there exists $f^* \in L^p$ with $f^*(S(x)) = f^*$ such that

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} f(S^k(x)) - f^*(x) \right\|_p \to 0, \quad n \to \infty \tag{16}$$

We can then extend Birkhoff's ergodic theorem as follows: If $S : X \to X$ is measure preserving and ergodic. Then for any integrable function f, the average of f along the trajectory of S is equal almost everywhere to the average of f over the space X

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(S^k(x)\right) = \frac{1}{\mu(X)} \int_X f(x)\mu(dx) \tag{17}$$

This theorem also states in other words that any function f in the L^p space can be obtained by sampling via an ergodic dynamical system. A further generalisation and extension of Birkhoff's ergodic theorem can also be seen in the Wiener-Winter theorem.

In general, the measure of any dynamical system is not a known, there are few exceptions however e.g. Hamiltonian system. It is easier to compute for spectral objects (eigenvalues, eigenfunction, and eigenspaces) of operators by sampling along trajectories and relying on Birkhoff's ergodic theorem for convergence. Extensions beyond L^p space and non-measure preserving (e.g. dissipative) transformations S are current topics of research.

Current research

Since the Perron-Frobenius and the Koopman operator are adjoint to each other, theoretically it does not matter which tools we use to study a dynamical system's behavior. However, different methods have been developed for the numerical approximation of these two operators.

As we have described earlier, the function space of the Frobenius-Perron Operator is in L^1 therefore it is theoretically limited to solving low-dimensional problems. The classical tools used for approximating the Frobenius-Perron operator is the Ulam-Galerkin methods or also called as Generalised Galerkin methods [3]. The numerical approximation of the Frobenius-Perron Operator typically requires short simulations for a large number of initial conditions which grows exponentially with the number of dimensions. The approximation of the Koopman operator typically requires longer simulations but fewer intial conditions [4]. The need for a large number of initial conditions implies that we already know a lot of things about the dynamic system of which we are analysing. Hence, is not very useful to apply it to systems that we know nothing about. Due to a number of downsides both theoretically and numerically, the Frobienus-Perron Operator are usually confined to very specific type of problems. The Koopman Operator holds a larger promise of solving a larger class of problems, therefore it would be the main focus of this literature review.

The work of [14] laid the ground work for the analysis of measure preserving deterministic or stochastic dynamical systems using Koopman Operators. The work introduces a way to reduce the dimensionality of the dynamical system, by finite-dimensional projections on to Koopman eigenspaces. The Hilbert space of L^2 functions are considered in this work.

A lot of research has been done to extend this result to non-conservative systems that admit an attractor (non-measure preserving system). Attempts to solve this has been to introduce more appropriate spaces of functions such as the spaces of continously differentiable functions [11], spaces of analytic functions [10], or generalised Hardy spaces [17] [13]

Another work [15] attempts to achieve a stronger result and solve the theoretical issues of Koopman operator theory beyond L^2 spaces. The work generalises the Koopman eigenfunctions over dynamical systems with globally stable attractors and defining a new class of Hilbert spaces of functions that can capture its dynamics. The new class of Hilbert spaces introduced are the Modulated Fock Spaces (Fock-Bargmann Space) and the Averaging Kernel Hilbert Space (AKHS) which is the modification of the RKHS.

It was not until recently that it was discovered that the Dynamic mode Decomposition (DMD) [18] which had its roots in fluid mechanics, can be used to approximate the Koopman eigenfunction leading to a considerable spike of interest in the field. The data-driven approximation methods can be separated into two main methods - finite-dimensonl matrix approximation of Koopman operator (DMD) and generalised Laplace averages (GLA). GLA methods differ from DMD that they do not provide an approximation of the operator but they first seek an approximation to eigenvalues then use projection theorems to obtain eigenfunctions and modes [2].

Data-driven approximations of the Koopman eigenfunctions have been applied to solve Control problems [7] [6] and many other system identification problems in other fields [9]. Convex formulations utilising Koopman eigenfunctions has been introduced in [16] [8] to solve prediction and control problems (LQR and MPC).

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